NPTEL Video Course
Advanced Complex Analysis – Part 2: Singularity at Infinity, Infinity as a Value, Compact Spaces of Meromorphic Functions for the Spherical Metric and Spherical Derivative, Local Analysis of Normality, Theorems of Marty-Zalcman-Montel-Picard-Royden-Schottky

http://nptel.ac.in/syllabus/111106094/

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Mid-Course Exam (Syllabus: Units 1 to 8) Time: Two Hours Maximum Marks: 40

1. State the generalised version of Liouville's theorem. 2 **marks**

2. Consider the function
$$f(z) = \frac{z^2 - 2z + 3}{z^3 + 1}.$$
   a) What kind of a singular point is ∞ for f? Why?
   b) Write out the singular (principal) and analytic parts of f at ∞.
   c) Verify the Residue Theorem for the extended complex plane for f.

   7 **marks**

3. Show that $f_n(z) = z^{-n}$ converges normally to ∞ in the unit disc $|z| < 1$. Is the convergence uniform? Justify your answer. 5 **marks**

4. Can a sequence of holomorphic (analytic) functions converge normally in the spherical metric to a strictly meromorphic function? Why? 2 **marks**

5. What kind of singularity does $f(z) = e^z$ have at ∞? Why? 3 **marks**

6. A function $f(z)$ has an isolated singularity at z_0. Given that f is a one-to-one mapping in a neighborhood of z_0, what kind of singularity can z_0 be? Why? 3 **marks**

7. State the Casorati-Weierstrass Theorem. Show that the only one-to-one entire functions onto the complex plane are of the form $f(z) = az + b, a \neq 0, b \in \mathbb{C}$. 6 **marks**

8. Let $f(z) = (z^2 + 1)^{-1}$.
   a) Find the spherical derivatives $f^{\#}(0)$ and $f^{\#}(i)$.
   b) Identify the extended complex plane with the Riemann sphere under the stereographic projection. Find the arc length of $f(\{z : |z| = 1\})$. 7 **marks**

9. Let $f(z)$ have a pole at z_0. Prove that $f^{\#}(z_0) = (1/f)^{\#}(z_0)$. 5 **marks**